DYNAMICS OF THE THERMOHALINE CIRCULATION UNDER UNCERTAINTY

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Abstract. The ocean thermohaline circulation under uncertainty is investigated by a random dynamical systems approach. It is shown that the asymptotic dynamics of the thermohaline circulation is described by a random attractor and by a system with finite degrees of freedom.

1. Introduction

The ocean thermohaline circulation(THC) involves water masses sinking at high latitudes and upwelling at lower latitudes. The process is maintained by water density contrasts in the ocean, which themselves are created by atmospheric forcing, namely, heat and freshwater exchange via evaporation and precipitation at the air-sea interface. Thus the ocean THC is driven by fluxes of heat and freshwater through the air-sea interface. During the THC, water masses carry heat or cold around the globe. Thus, it is believed that the global ocean THC plays an important role in the climate [24].

The formulation and analysis of mathematical models is central to the progress of better understanding of the THC dynamics and its impact on climate change. Apart from a detailed modeling of the climate system using coupled general circulation models, sometimes simplified climate models could give insight into the general characteristics of the climate system. The most simplified climate models can be described in terms of stochastic differential equations [16, 4, 17]. These stochastic climate models can be viewed as comprehensive paradigms or metaphors for particular features of the climate system.

We consider a two-dimensional thermohaline ocean circulation model in the latitude-depth (meridional) plane, in terms of the stochastic Navier-Stokes fluid equations (vorticity form) and the transport equations for heat and salinity, together with air-sea flux or Neumann boundary conditions. The

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We intend to investigate the characteristics of the THC's dynamics when some random effect is taken into account. Our approach here is a random dynamical systems approach [2].

This paper is organized as follows. In the next section we present the THC model, and discuss the well-posedness of this model in Section 3. Section 4 is devoted to the investigation of the dynamical behavior of this model: random attractor and finite dimensionality.

2. A Model for the Thermohaline Circulation

We consider the ocean thermohaline circulation in a bounded domain, i.e., a square

$$D = \{(y,z): \ -l \le y \le l, \ 0 \le z \le d\},$$

on the meridional, latitude-depth (y,z)-plane, as used by various authors [20, 21, 27, 5, 12]. It is composed of the Boussinesq version of the Navier-Stokes equations for oceanic fluid velocity $(v(y,z,t), w(y,z,t))$ and transport equations for the oceanic salinity $S(y,z,t)$ and the oceanic temperature $T(y,z,t)$ in dimensional form:

$$\begin{aligned}
v_t + vv_y + wv_z &= -p_y + \nu\Delta v + \text{noise} \ , \\
w_t + vw_y + ww_z &= -p_z - g(\alpha_S S - \alpha_T T) + \nu\Delta w + \text{noise} \ , \\
v_y + w_z &= 0, \\
T_t + vT_y + wT_z &= \kappa_T \Delta T, \\
S_t + vS_y + wS_z &= \kappa_S \Delta S,
\end{aligned}$$

where α_S and α_T are the coefficients of volume expansion for salt and heat, respectively; g is the gravitational acceleration; ν is the viscosity; and κ_S and κ_T are salt and heat diffusivities, respectively. The density is $\rho = \rho_0(1 + \alpha_S S - \alpha_T T)$ with ρ_0 the mean sea water density. The noise in the Navier-Stokes equations is due to various fluctuations such as random wind stress forcing [18, 28, 15]. Presumably, the noise also affects the transport of heat and salinity to some extent, but we will ignore this effect.

As discussed in [27, 20], this may be regarded as a zonally averaged model of the world ocean. The effect of the rotation can be parameterized in the magnitude of the viscosity and diffusivity terms. Introducing the stream function $\psi(y,z,t)$ for the velocity field,

$$v = -\psi_z, \ \ w = \psi_y,$$

we can rewrite the above model in the vorticity form with only three unknowns ψ, T, S:

$$\Delta\psi_t + J(\psi, \Delta\psi) = g(\alpha_T T_y - \alpha_S S_y) + \nu\Delta^2\psi + \dot{\mathcal{W}}_1 \ , \tag{1}$$

$$T_t + J(\psi, T) = \kappa_T \Delta T, \tag{2}$$

$$S_t + J(\psi, S) = \kappa_S \Delta S, \tag{3}$$

where $\mathcal{W}_1(y, z, t)$ is a Wiener process defined on a underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be specified below. The noise is described by the generalized time-derivative of the Wiener process. The fluctuating noise in the oceanic fluid equation is usually of a shorter time scale than the response time scale of the large scale oceanic THC. We thus assume the noise is white in time (uncorrelated in time) but it is allowed to be colored in space, i.e., it may be correlated in space variables. Note that the pressure field p is eliminated in the vorticity form of the Navier-Stokes equations.
Boundary conditions for the oceanic fluid are no normal flow and free-slip (no-stress) on the whole boundary [19]:

$$\psi = \Delta\psi = 0 \ \text{ on } \partial D. \tag{4}$$

The boundary conditions for temperature and salinity are Neumann type. On the air-sea interface $z = d$, the heat/temperature flux and freshawater/salinity flux are prescribed as

$$T_z(y, d) = \lambda(\theta(y) - T), \quad S_z(y, d) = F(y). \tag{5}$$

Here $\theta(y)$ is the prescribed (known) atmosphere surface temperature and $F(y)$ is the mean freshwater flux (known). Moreover, $\lambda = \frac{B_T}{\rho_0 C_p \kappa_T}$, with B_T being the surface exchange coefficient of heat and C_p the heat capacity.
Zero flux boundary conditions are taken for T and S on fluid bottom ($z = 0$) and on fluid side ($y = \pm l$):

$$T_z(y, 0) = 0, \ S_z(y, 0) = 0, \tag{6}$$

$$T_y(\pm l, z) = 0, \ S_y(\pm l, z) = 0. \tag{7}$$

The THC model above involves stochastic and deterministic partial differential equations (PDEs) and Neumann boundary conditions.

3. Well-Posedness

In this section we will show that (1)-(3) defines a well-posed model. First we introduce some function spaces from the theory of partial differential equations.
Let $W_2^1(D)$ be the Sobolev space of function on D with the first generalized derivative in $L_2(D)$, the function space of square integrable functions on D with norm and inner product

$$||u||_{L_2} = (\int_D |u(x)|^2 dD)^{\frac{1}{2}}, \quad (u, v)_{L_2} = \int_D u(x)v(x)dD, \quad u, v \in L_2(D)$$

The space $W_2^1(D)$ is equipped with the norm

$$||u||_{W_2^1} = ||u||_{L_2} + ||\partial_y u||_{L_2} + ||\partial_z u||_{L_2}$$

Let the $\overset{\circ}{W}{}_2^1(D)$ be the space of functions vanishing on the boundary ∂D of D. The norm of this space is defined as

$$\|u\|_{\overset{\circ}{W}{}_2^1(D)} = \|\partial_y u\|_{L_2} + \|\partial_z u\|_{L_2}. \tag{8}$$

Similarly, we can define function spaces on the interval $(0, d)$ denoted by $L_2(0, d)$ and $W_2^1(0, d)$.
Another Sobolev space is $\dot{W}_2^1(D)$ which is a subspace of $W_2^1(D)$ consisting of functions u with zero mean: $\int_D u dD = 0$. A norm equivalent to the W_2^1-norm on $\dot{W}_2^1(D)$ is given by the right hand side of (8). For functions in $L_2(D)$ having the same property, we write as $\dot{L}_2(D)$.
There exists a *continuous trace operator*:

$$\gamma_{\partial D} : W_2^1(D) \to H^{\frac{1}{2}}(\partial D).$$

Here $H^{\frac{1}{2}}(\partial D)$ is a boundary space, see Adams [1] Chapter 7 or below. Similarly, we can introduce trace operators that map onto a part of the boundary of ∂D for instance for the subset $\{(y, z) \in \bar{D} | z = d\}$ of $\bar{D}$. For this mapping we will write

$$\gamma_{z=d} : W_2^1(D) \to H^{\frac{1}{2}}(0, d). \tag{9}$$

The adjoint operator

$$\gamma_{z=d}^* : (H^{\frac{1}{2}}(0, d))' \to (W_2^1(D))'$$

is also continuous, where $'$ denotes the dual space for a given Banach space. If set vorticity $q = \Delta\psi$ we can homogenize boundary conditions to obtain:

$$q_t + J(\psi, q) = g(\alpha_T T_y - \alpha_S S_y) + \nu\Delta q + \dot{\mathcal{W}}_1 \, , \tag{10}$$

$$T_t + J(\psi, T) = \kappa_T \Delta T + \gamma_{z=d}^*(\lambda(\theta(y) - \gamma_{z=d} T)), \tag{11}$$

$$S_t + J(\psi, S) = \kappa_S \Delta S + \gamma_{z=d}^* F(y). \tag{12}$$

New homogeneous boundary conditions are:

$$\begin{aligned} \psi = 0, \quad q = 0 \ \text{ on } \partial D, \\ T_z(y, d) = 0, \quad S_z(y, d) = 0, \\ T_z(y, 0) = 0, \quad S_z(y, 0) = 0, \\ T_y(\pm l, z) = 0, \quad S_y(\pm l, z) = 0. \end{aligned}$$

For convenience, we introduce the vector notation for unknown geophysical quantities

$$u = (q, T, S). \tag{13}$$

Now we can define the linear differential operator from (10)-(12)

$$Au = \begin{pmatrix} -\nu\Delta q \\ -\kappa_T \Delta T \\ -\kappa_S \Delta S \end{pmatrix}.$$

We assume that F and $\theta \in L_2(0,d)$. Note that

$$\frac{d}{dt}\int_D S dydz = \int_0^d F(y)dy = constant.$$

It is reasonable (see [12]) to assume that

$$\int_0^d F(y) = 0$$

and thus $\int_D S dydz$ is constant in time and we may assume it is zero:

$$\int_D S dydz = 0.$$

Thus we have the usual Poincaré inequality for S. However, this is not true for T. Fortunately we can derive the following Poincaré inequality for T

$$\|T\|^2 \le c(\Omega)(\|\gamma_{z=1}T\|_{L^2}^2 + \|\nabla T\|_{L_2}^2), \tag{14}$$

as in Temam [26] where $c(\Omega)$ is a constant dependent on Ω.
Introduce the phase space for our system $H = L_2(D) \times L_2(D) \times \dot{L}_2(D)$ with the usual L_2 inner product and $V = \mathring{W}_2^1(D) \times W_2^1(D) \times \dot{W}_2^1(D)$.
It is obvious that the linear operator $A : V \to V'$ is positive definite. And define the nonlinear operator $G(u) := G_1(u) + G_2(u)$ where

$$G_1(u)[y,z] = \begin{pmatrix} -J(\psi, q) \\ -J(\psi, T) \\ -J(\psi, S) \end{pmatrix}[y,z].$$

and

$$G_2(u)[y,z] = \begin{pmatrix} g(\alpha_T T_y - \alpha_S S_y) \\ \gamma_{z=d}^*(\lambda(\theta(y) - \gamma_{z=d}T)) \\ \gamma_{z=d}^* F(y) \end{pmatrix}[y,z].$$

Then the THC system can be rewritten as a stochastic differential equation on V':

$$\frac{du}{dt} + Au = G(u) + \dot{W} \qquad u(0) = u_0 \in H, \tag{15}$$

where $W = (\mathcal{W}_1, 0, 0)$, $\dot{W}$ is a white noise as the generalized temporal derivative of a Wiener process W with continuous trajectories on $\mathbb{R}$ and with values in $L_2(D)$. It is sufficient for this regularity that the trace of the covariance is finite with respect to the space $L_2(D)$: $\mathrm{tr}_{L_2} Q < \infty$. In particular, we can choose the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the set of elementary events Ω consists of the paths of W and the probability measure $\mathbb{P}$ is the Wiener measure with respect to covariance Q.

Through integration by parts and direct estimation or from [6] we have the following lemmas.

Lemma 3.1. *The operator $G_1 : V \to H$ is continuous. In particular, we have*

$$\langle G_1(u), u\rangle = 0.$$

and

Lemma 3.2. *The following estimation holds*

$$\|G_2(u)\|_{V'} \leq c_1\|u\|_V + c_2.$$

for some positive constants c_1, c_2.

In the following we need a stationary Ornstein-Uhlenbeck process solving the linear stochastic equation on D

$$\frac{d\eta}{dt} - \nu(k+1)\Delta\eta = \dot{\mathcal{W}}_1 \tag{16}$$

with the homogeneous Neumann boundary condition at ∂D. Here k can be seen as a very large controlling parameter.

Lemma 3.3. *Suppose that the covariance Q has a finite trace : $\mathrm{tr}_{L_2}Q < \infty$. Then (16) has a unique stationary solution generated by*

$$(t, \omega) \to \eta(\theta_t\omega).$$

Moreover, η is a tempered random variable in $W_2^1(D)$ and has trajectories in the space $L^2_{loc}(\mathbb{R}; W_2^3(D))$. Then $Z(\omega) = (\eta(\omega), 0, 0)$ is a random variable in V.

For the proof we refer to [9].

If we set

$$(\tilde{q}, T, S) = v := u - Z = (q - \eta, T, S), \tag{17}$$

then we obtain a random differential equation in V'

$$\frac{dv}{dt} + Av = G_1(v) + \tilde{G}_2(v + Z(\theta_t\omega)), \qquad v(0) = v_0 \in H, \tag{18}$$

where $\tilde{G}_2 = G_2 + (J(\eta, \tilde{q}) + J(\Delta^{-1}\tilde{q}, \Delta\eta) + J(\eta, \Delta\eta) - \nu k\Delta\eta, 0, 0)$.

The above equation (18) is a differential equation with random coefficients then it can be treated sample-wise for any sample ω. We are looking for solution v in

$$C([0, \tau]; H) \cap L_2(0, \tau; V),$$

for all $\tau > 0$. If we can solve this equation then $u := v + Z$ defines a solution version of (15). For the well posedness of the problem we now have the following result.

Theorem 3.4. (**Well-Posedness**) *For any time $\tau > 0$, there exists a unique solution of (18) in $C([0,\tau];H)\cap L_2(0,\tau;V)$. In particular, the solution mapping*

$$\mathbb{R}^+ \times \Omega \times H \ni (t,\omega,v_0) \to v(t) \in H$$

is measurable in its arguments and the solution mapping $H \ni v_0 \to v(t) \in H$ is continuous.

Proof. By the properties of A and G_1 (see Lemma 3.1), the random differential equation (18) is essentially similar to the 2-dimensional Navier Stokes equation. Note that $\tilde{G}_2$ is only an affine mapping. Hence we have existence and uniqueness and the above regularity assertions. □

Now we can define a random dynamical system since the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t,\omega,v_0) \to v(t,\omega,v_0) =: \varphi(t,\omega,v_0) \in H.$$

is well defined. First we define a so-called metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. $\theta_t : \Omega \to \Omega, t \in \mathbb{R}$ is a family of measure preserving transformations such that $(t,\omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = id, \theta_{t+s} = \theta_t\theta_s$, for all $s,t \in \mathbb{R}$. Furthermore, the shift θ_t is ergodic if we define it as

$$w(\cdot,\theta_t\omega) = w(\cdot + t,\omega) - w(t,\omega) \quad \text{for } t \in \mathbb{R}$$

which is called the *Wiener shift*. A random dynamical system $\varphi(t,\omega,u)$ is well defined now since the cocycle property

$$\begin{aligned}&\varphi(t+\tau,\omega,u) = \varphi(t,\theta_\tau\omega,\varphi(\tau,\omega,u)) \quad \text{for } t,\, \tau \geq 0\\ &\varphi(0,\omega,u) = u\end{aligned}$$

for any $\omega \in \Omega$ and $u \in H$. For more detail about random dynamical systems we refer to [2].

4. Random dynamics

In this section, we investigate random dynamics of the THC. First we are going to show that the random THC model is dissipative, i.e., it has an random absorbing set in the following sense:

Definition 4.1. *A random set $B = \{B(\omega)\}_{\omega\in\Omega}$ consisting of closed bounded sets $B(\omega)$ is called absorbing for a random dynamical system φ if we have for any random set $D = \{D(\omega)\}_{\omega\in\Omega}$, $D(\omega) \in H$ bounded, such that $t \to \sup_{y\in D(\theta_t\omega)}\|y\|_H$ has a subexponential growth for $t \to \pm\infty$*

$$\begin{aligned}&\varphi(t,\omega,D(\omega)) \subset B(\theta_t\omega) \quad \textit{for } t \geq t_0(D,\omega)\\ &\varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega)) \subset B(\omega) \quad \textit{for } t \geq t_0(D,\omega).\end{aligned} \tag{19}$$

B is called forward invariant if

$$\varphi(t,\omega,u_0) \in B(\theta_t\omega) \quad \textit{if } u_0 \in B(\omega) \quad \textit{for } t \geq 0.$$

Consider THC system separately. For this we introduce the spaces

$$\tilde{H} = L_2(D) \times \dot{L}_2(D),$$
$$\tilde{V} = W_2^1(D) \times \dot{W}_2^1(D).$$

We choose a subset of dynamical variables of our system (10)-(12)

$$\tilde{v} = (T, S). \tag{20}$$

Applying the chain rule to $\|\tilde{v}\|_H^2$, we obtain by Lemma 3.1

$$\begin{aligned}&\frac{d}{dt}\|\tilde{v}\|_{\tilde{H}}^2 + 2\kappa_T\|\nabla T\|_{L_2}^2 + 2\kappa_S\|\nabla S\|_{L_2}^2 \\ =&2\lambda(\theta(y), \gamma_{z=d}T)_{L_2} - 2\lambda(\gamma_{z=d}T, \gamma_{z=d}T)_{L_2} + 2(F(y), \gamma_{z=d}S)_{L_2}.\end{aligned} \tag{21}$$

Cauchy-Schwarz inequality yields the following estimates

$$2\lambda(\theta(y), \gamma_{z=d}T)_{L_2} - 2\lambda(\gamma_{z=d}T, \gamma_{z=d}T)_{L_2} \leq \frac{\lambda}{a}\|\theta(y)\|_{L_2}^2 + (a-2)\lambda\|\gamma_{z=d}T\|_{L_2}^2,$$

where a is a positive constant that satisfies $2 > a > 2 - \frac{2\kappa_T}{\lambda}$. Applying the trace theorem $\|\gamma_{z=d}S\|_{L_2}^2 \leq c_3\|\gamma_{z=d}S\|_{H^{\frac{1}{2}}} \leq c_3\|S\|_{W_2^1}$ and for any $\varepsilon > 0$ we can find a $c_4(\varepsilon)$ such that

$$2(F(y), \gamma_{z=d}S)_{L_2} \leq \varepsilon\|S\|_{W_2^1}^2 + c_4(\varepsilon)\|F(y)\|_{L_2}^2,$$

for some ε.
Thus by using the Poincaré inequality for $S \in \dot{W}_2^1(D)$, (14) and choosing ε small enough, we conclude

$$\frac{d}{dt}\|\tilde{v}\|_{\tilde{H}}^2 + \alpha(\|\nabla\tilde{v}\|_{\tilde{H}}^2 + \|\tilde{v}\|_{\tilde{H}}^2) \leq c_5 \tag{22}$$

where α, c_5 is determined by λ, κ_T, κ_S, $\|\theta\|_{L_2}$, $\|F(y)\|_{L_2}$ and $\nabla\tilde{v}$ is defined by $(\nabla_{y,z}T, \nabla_{y,z}S)$.

By the Gronwall inequality, we finally conclude that

$$\|\tilde{v}\|_{\tilde{H}}^2 \leq \|\tilde{v}(0)\|_{\tilde{H}}^2 e^{-\alpha t} + \frac{c_5}{\alpha}. \tag{23}$$

Then we can easily see that the ball $B(0, R_1)$, where $R_1^2 = \frac{2c_5}{\alpha}$, absorbs $\tilde{v}$ in the sense of definition 4.1.

To prove the dissipativity of the dynamical system φ we have to study $\|\tilde{q}\|_{L_2}$. From (18), we have

$$\tilde{q}_t = -J(\psi, q) + g(\alpha_T T_y - \alpha_S S_y) + \nu\Delta\tilde{q} - \nu k\Delta\eta. \tag{24}$$

Then

$$\frac{1}{2}\frac{d}{dt}\|\tilde{q}\|_{L_2}^2 = (J(\psi,q),\eta) - \nu\|\nabla\tilde{q}\|_{L_2}^2 - \nu k(\Delta\eta,\tilde{q}) + g(\alpha_T T_y - \alpha_S S_y, \tilde{q}). \tag{25}$$

From the definition of $J(\cdot,\cdot)$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned}(J(\psi,q),\eta) = (J(\psi,\eta),q) &\le \|\nabla q\|_{L_2}\|q\|_{L_2}\|\nabla\eta\|_{L_2}\\ &\le \frac{\nu}{2}\|\nabla q\|_{L_2}^2 + \frac{1}{2\nu}\|\nabla\eta\|_{L_2}^2\|q\|_{L_2}^2\\ &\le \frac{\nu}{2}\|\tilde{q}\|_{W_2^1}^2 + \frac{\nu}{2}\|\eta\|_{W_2^1}^2 + \frac{1}{2\nu}\|\eta\|_{W_2^1}^2\|\tilde{q}\|_{L_2}^2 + \frac{c_6}{8\nu}\|\eta\|_{W_2^1}^4.\end{aligned}$$

Here c_6 is the constant in the Poincaré inequality for $\eta \in \overset{\circ}{W}{}_2^1\,(0,d)$. And for any $\varepsilon > 0$ we can find $c_7(\varepsilon) > 0$ and $c_8(\varepsilon) > 0$ such that

$$\begin{aligned}g(\alpha_T T_y - \alpha_S S_y, \tilde{q}) &\le \tilde{g}\|\nabla\tilde{v}\|_{\tilde{H}}\|\tilde{q}\|_{L_2}\\ &\le \frac{\varepsilon}{4}\|\tilde{q}\|_{L_2}^2 + c_7(\varepsilon)\tilde{g}^2\|\nabla v\|_{\tilde{H}}^2,\end{aligned}$$

$$\begin{aligned}-\nu k(\Delta\eta,\tilde{q}) &= \nu k(\nabla\eta,\nabla\tilde{q})\\ &\le \frac{\varepsilon}{4\lambda_1}\|\tilde{q}\|_{W_2^1}^2 + \lambda_1 c_8(\varepsilon)k^2\nu^2\|\eta\|_{W_2^1}^2,\end{aligned}$$

where $\tilde{g}$ is determined by α_T, α_S and g, λ_1 is the first eigenvalue of the operator $-\Delta$ on $(0,d)$ with Neumann boundary condition.
Collecting all these estimates, we have

$$\begin{aligned}\frac{d}{dt}\|\tilde{q}\|_{L_2}^2 &\le -\gamma(\theta_t\omega)\|\tilde{q}\|_{L_2}^2 + \delta(\varepsilon)\|\nabla\tilde{v}\|_{\tilde{H}}^2 + r(\theta_t\omega)\\ &\le -\gamma(\theta_t\omega)\|\tilde{q}\|_{L_2}^2 + r(\theta_t\omega) + \frac{\delta(\varepsilon)c_5}{\alpha} - \frac{\delta(\varepsilon)}{\alpha}\frac{d}{dt}\|\tilde{v}\|_{\tilde{H}}^2,\end{aligned}$$

where $\gamma(\omega) = \lambda_1\nu - \varepsilon - \frac{1}{\nu}\|\eta\|_{W_2^1}^2$, $\delta(\varepsilon) = c_7(\varepsilon)\tilde{g}^2$ and

$$r(\omega) = (2\lambda_1 c_8(\varepsilon)k^2\nu^2 + \nu)\|\eta\|_{W_2^1}^2 + \frac{c_6}{4\nu}\|\eta\|_{W_2^1}^4. \tag{26}$$

Then Gronwall inequality yields

(27)

$$\begin{aligned}
\|\tilde{q}(t,\omega,u_0)\|_{L_2}^2 \le{}& \|v_0\|_H^2 e^{-\int_0^t \gamma(\theta_s\omega)ds} \\
&+ \int_0^t \Big(\frac{\delta(\varepsilon)c_5}{\alpha} + r(\theta_s\omega) - \frac{\delta(\varepsilon)}{\alpha}\frac{d}{ds}\|\tilde{v}(s)\|_{\tilde{H}}^2\Big) e^{-\int_s^t \gamma(\theta_\tau\omega)d\tau} ds \\
\le{}& \|v_0\|_H^2 e^{-\int_0^t \gamma(\theta_s\omega)ds} \\
&+ \int_0^t \Big(\frac{\delta(\varepsilon)c_5}{\alpha} + r(\theta_s\omega)\Big) e^{-\int_s^t \gamma(\theta_\tau\omega)d\tau} ds \\
&+ \frac{\delta(\varepsilon)}{\alpha}\int_0^t \|\tilde{v}\|_{\tilde{H}}^2 \gamma(\theta_s\omega) e^{-\int_s^t \gamma(\theta_\tau\omega)d\tau} ds + \frac{\delta(\varepsilon)}{\alpha}\|\tilde{v}(0)\|_{\tilde{H}}^2 e^{-\int_0^t \gamma(\theta_s\omega)ds}.
\end{aligned}$$

We will show that the right hand of (27) is finite as $t \to \infty$. In fact we have

Lemma 4.2. *If the controlling parameter k is large enough that*

$$\lambda_1 > \frac{tr_{L_2}Q}{(k+1)\nu^3}$$

and $\varepsilon < \frac{\lambda_1\nu}{2}$ then

$$E\gamma(\omega) > 0.$$

Proof. Itô formula applied to $\|\eta\|_{L_2}^2$ yields

$$\|\eta(\theta_\tau\omega)\|_{L_2}^2 + 2(k+1)\nu\int_0^\tau \|\eta(\theta_s\omega)\|_{W_2^1}^2 ds = \|\eta(\omega)\|_{L_2}^2 + 2\int_0^\tau (\eta, dw)_{L_2} + \tau tr_{L_2}Q.$$

Hence we can easily get that $E\|\eta\|_{W_2^1}^2 \le \frac{\lambda_1\nu^2}{2}$. Then $E\gamma(\omega) > 0$. □

Now we can estimate the $\|\tilde{q}\|_{L_2}^2$. First we have

$$\lim_{t\to\infty}\Big(\|v_0\|_H^2 + \frac{\delta(\varepsilon)}{\alpha}(c_5 t + \|\tilde{v}(0)\|_{\tilde{H}}^2)\Big) e^{-\int_0^t \gamma(\theta_s\omega)ds} = 0, \quad P.a.s.$$

And note that $\|\tilde{v}\|_{\tilde{H}}^2$ is bounded by $\|\tilde{v}(0)\|_{\tilde{H}}^2 e^{-\alpha t} + \frac{c_5}{\alpha}$ which tends to $\frac{c_5}{\alpha}$ exponentially. We replace ω by $\theta_{-t}\omega$ to construct the radius of the absorbing set. Then we have

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t \Big(r(\theta_{t-s}\omega) + \frac{\delta(\varepsilon)}{\alpha}\|\tilde{v}\|_{\tilde{H}}^2 \gamma(\theta_{t-s}\omega)\Big) e^{-\int_s^t \gamma(\theta_{t-\tau}\omega)d\tau} ds \\
={}& \lim_{t\to\infty}\int_{-t}^0 \Big(r(\theta_s\omega) + \frac{\delta(\varepsilon)}{\alpha}\|\tilde{v}\|_{\tilde{H}}^2 \gamma(\theta_s\omega)\Big) e^{-\int_s^t \gamma(\theta_\tau\omega)d\tau} ds \\
=:{}& R_2^2(\omega) < \infty.
\end{aligned}$$

Collecting all the estimates we have

Lemma 4.3. *The random set $\{B(\omega)\}_{\omega\in\Omega}$ given by closed balls $B(0, R(\omega))$ in H with center zero and radius $R^2(\omega) := R_1^2 + R_2^2(\omega)$ is an absorbing and forward invariant set for the random dynamical system φ generated by (18).*

For the application in the following we need the particular regularity of the absorbing set. To this end we introduce the function space

$$\mathcal{H}^s := \{u \in H : \|u\|_s^2 := \|A^{\frac{s}{2}}u\|_H^2 < \infty\}$$

where $s \in \mathbb{R}$. The operator A^s is the s-th power of the positive and symmetric operator A. Note that these spaces are embedded into the Sobolev spaces H^s, $s > 0$. The norm of these spaces is denoted by $\|\cdot\|_{H^s}$.

For our aim we show that $v(1, \omega, D)$ is a bounded set in $\mathcal{H}^s$ for some $s > 0$. Consider $t\|v(t)\|_s^2$ and by chain rule we have

$$\frac{d}{dt}(t\|v(t)\|_s^2) = \|v(t)\|_s^2 + t\frac{d}{dt}\|v(t)\|_s^2.$$

The second term in the above formula can be expressed as follows

$$\begin{aligned} t\frac{d}{dt}(A^{\frac{s}{2}}v, A^{\frac{s}{2}}v)_H =& 2t(\frac{d}{dt}v, A^s v)_H = -2t(Au, A^s v)_H + 2t(G_1(v), A^s v)_H \\ &+ 2t(\tilde{G}_2(v + Z(\theta_t\omega)), A^s v)_H. \end{aligned}$$

Notice that for $\varepsilon > 0$ there are constants $c_9 - c_{14}$ such that

$$(J(\eta, \tilde{q}), A^s v)_H \leq c_9\|v\|_V\|\eta\|_{1+s}\|v\|_{1+s} \leq c_{10}(\varepsilon)\|\eta\|_{1+s}\|v\|_V^2 + \varepsilon\|v\|_{1+s}^2,$$

$$(J(\psi, \Delta\eta), A^s v)_H \leq c_{11}\|\Delta\eta\|_1\|\psi\|_{1+s}\|v\|_{1+s} \leq c_{12}(\varepsilon)\|\eta\|_2^2\|v\|_{L_\infty(0,T;H)}^2 + \varepsilon\|v\|_{1+s}^2,$$

$$(J(\eta, \Delta\eta), A^s v)_H \leq c_{13}\|\Delta\eta\|_1\|\eta\|_{1+s}\|v\|_{1+s} \leq c_{14}(\varepsilon)\|\eta\|_2^2\|\eta\|_{1+s}^2 + \varepsilon\|v\|_{1+s}^2,$$

and

$$\int_0^t \|v\|_s^2 ds \leq c_{15}\int_0^t \|v\|_V^2 ds, \quad s \leq 1$$

for the embedding constant c_{15} between $\mathcal{H}^s$ and V. Then by the similar arguments as in [6] or [13] we have the following dissipative property of THC.

Lemma 4.4. *For the random dynamical system φ generated by (18), there exists a compact random set $B = \{B(\omega)\}_{\omega\in\Omega}$ which satisfies Definition 4.1.*

We define

$$B(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(0, R(\theta_{-1}\omega)))} \subset \mathcal{H}^s, \quad 0 < s < \frac{1}{4}. \tag{28}$$

In particular, $\mathcal{H}^s$ is compactly embedded in H.

Now we show that the dynamics of THC is described by a random attractor. First we recall the following basic concept; See [7].

Definition 4.5. *Let φ be a random dynamical system. A random set $A = \{A(\omega)\}_{\omega\in\Omega}$ consisting of compact nonempty sets $A(\omega)$ is called random global attractor if for any random bounded set D we have for the limit in probability*

$$(\mathbb{P}) \lim_{t\to\infty} \mathrm{dist}_H(\varphi(t,\omega,D(\omega)), A(\theta_t\omega)) = 0$$

and

$$\varphi(t,\omega,A(\omega)) = A(\theta_t\omega)$$

for any $t \geq 0$ and $\omega \in \Omega$.

The following theorem [7] gives a condition of the existence of random attractor.

Theorem 4.6. *Let φ be a random dynamical system on the state space H which is a separable Banach space such that $x \to \varphi(t,\omega,x)$ is continuous. Suppose that B is a set ensuring the dissipativity given in Definition (4.1). In addition, B has a subexponential growth (see Definition (4.1)) and is regular (compact). Then the dynamical system φ has a random attractor.*

Then from the above analysis we have the following result to the random system φ generated by (18), and, via the transformation (17), thus to the original stochastic THC model.

Theorem 4.7. *(**Random Attractor**) The THC model (1)-(3) has a random attractor in the phase space $H = L_2(D) \times L_2(D) \times L_2(D)$.*

It is well known that, the global attractor for certain deterministic infinite-dimensional systems has finite dimension [26]. This result leads to the fact that asymptotic behavior of these systems can be described using finite-dimensional systems. And the similar theory has been developed for random dynamical systems; see, for instance, [8, 11, 23].
However, for the THC system we will apply another approach to prove the fact that the random attractor of (1)-(3) has only finitely many degrees of freedom. Namely we will use the concept of *determining functionals in probability* as in Chueshov et al. [6]. This concept was introduced by Foias and Prodi [14] for deterministic systems.

Definition 4.8. *We call a set $\mathcal{L} = \{l_j,\ j = 1,\cdots,N\}$ of linear continuous and linearly independent functionals on a space X continuously embedded in H (for instance $X = \mathcal{H}^s$ or V) asymptotically determining in probability if*

$$(\mathbb{P}) \lim_{t\to\infty} \int_t^{t+1} \max_j |l_j(\varphi(\tau,\omega,v_1) - \varphi(\tau,\omega,v_2))|^2 d\tau = 0$$

for two initial conditions $v_1,\ v_2 \in H$ implies

$$(\mathbb{P}) \lim_{t\to\infty} \|\varphi(t,\omega,v_1) - \varphi(t,\omega,v_2)\|_H = 0.$$

We introduce a constant $\varepsilon_{\mathcal{L}}$ to describe the qualitative difference of the space H and X for some set of functionals

$$\|u\|_H \leq C_{\mathcal{L}} \max_{l_i \in \mathcal{L}} |l_i(u)| + \varepsilon_{\mathcal{L}} \|u\|_X, \quad C_{\mathcal{L}} > 0. \tag{29}$$

We cite the following theorem from [6].

Theorem 4.9. *Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a set of linear continuous and linearly independent functionals on X. We assume that we have an absorbing and forward invariant set B in X such that for $\sup_{v \in B(\omega)} \|v\|_X^2$ the expression $t \to \sup_{v \in B(\theta_t \omega)} \|v\|_X^2$ is locally integrable and subexponentially growing. Suppose there exist a constant $c_{16} > 0$ and a measurable function $l \geq 0$ such that for v_1, $v_2 \in V$ we have for $\tilde{G}(\omega, v) = G_1(v) + \tilde{G}_2(v + Z(\omega))$*

$$\begin{aligned} &\langle -A(v_1 - v_2) + \tilde{G}(\omega, v_1) - \tilde{G}(\omega, v_2), v_1 - v_2 \rangle \\ &\quad \leq -c_{16} \|v_1 - v_2\|_V^2 + l(v_1, v_2, \omega) \|v_1 - v_2\|_H^2. \end{aligned}$$

Assume that

$$\frac{1}{m} \mathbb{E} \left\{ \sup_{v_1, v_2 \in B(\omega)} \int_0^m l(\varphi(t, \omega, v_1), \varphi(t, \omega, v_2), \theta_t \omega) dt \right\} < c_{16} \varepsilon_{\mathcal{L}}^{-2}$$

for some $m > 0$. Then $\mathcal{L}$ is a set of asymptotically determining functionals in probability for random dynamical system φ.

Now we go back to the random THC system. Note that for $\varepsilon > 0$ there are constants $c_{17}(\varepsilon), \cdots, c_{22}(\varepsilon)$ such that

$$\begin{aligned} &|\langle J(\tilde{\psi}_1, \tilde{q}_1) - J(\tilde{\psi}_2, \tilde{q}_2), \tilde{q}_1 - \tilde{q}_2 \rangle| = |\langle J(\tilde{\psi}_2 - \tilde{\psi}_1, \tilde{q}_1), \tilde{q}_2 - \tilde{q}_1 \rangle| \\ \leq &\frac{\varepsilon}{4} \|\tilde{q}_1 - \tilde{q}_2\|_V^2 + c_{17}(\varepsilon) \|\tilde{q}_1\|_{\overset{\circ}{W}{}_2^1(D)}^2 \|\tilde{q}_1 - \tilde{q}_2\|_{L_2}^2. \end{aligned}$$

Similarly

$$|\langle J(\tilde{\psi}_1, T_1) - J(\tilde{\psi}_2, T_2), T_1 - T_2 \rangle| \leq \frac{\varepsilon}{4} \|v_1 - v_2\|_V^2 + c_{18}(\varepsilon) \|T_1\|_{W_2^1(D)}^2 \|T_1 - T_2\|_{L_2}^2,$$

$$|\langle J(\tilde{\psi}_1, S_1) - J(\tilde{\psi}_2, S_2) S_1 - S_2 \rangle| \leq \frac{\varepsilon}{4} \|v_1 - v_2\|_V^2 + c_{19}(\varepsilon) \|S_1\|_{\dot{W}_2^1(D)}^2 \|S_1 - S_2\|_{L_2}^2,$$

$$|\langle \alpha_T T_y - \alpha_S S_y, \tilde{q}_1 - \tilde{q}_2 \rangle| \leq \frac{\varepsilon}{4} \|v_1 - v_2\|_V^2 + c_{20}(\varepsilon) \|\tilde{q}_1 - \tilde{q}_2\|_{L_2}^2,$$

and

$$|\langle J(\tilde{\psi}_1, \Delta\eta) - J(\tilde{\psi}_2, \Delta\eta), \tilde{q}_1 - \tilde{q}_2 \rangle| \leq c_{21} \|v_1 - v_2\|_V^2 \|\eta\|_{W_2^1}^2.$$

Then we have

$$\begin{aligned} &\langle -A(v_1 - v_2) + \tilde{G}(\omega, v_1) - \tilde{G}(\omega, v_2), v_1 - v_2 \rangle \\ \leq &- c_{22} \|v_1 - v_2\|_V^2 + l(v_1, v_2, \omega) \|v_1 - v_2\|_H^2. \end{aligned}$$

Here we can take k is large enough and ε is small enough such that $Ec_{22} = 1 - \varepsilon - c_{21}E\|\eta\|^2_{W_2^1} > 0$. And

$$l(v_1, v_2, \omega) = c_{17}(\varepsilon)\|\tilde{q}_1\|^2_{\overset{\circ}{W}{}_2^1(D)} + c_{18}(\varepsilon)\|T_1\|^2_{\overset{\circ}{W}{}_2^1(D)} + c_{19}(\varepsilon)\|S_1\|^2_{\overset{\circ}{W}{}_2^1(D)} + c_{20}(\varepsilon).$$

Now we set $X = \mathcal{H}^s, s \in (0, \frac{1}{4})$. In the above discussion we have shown that the set B, consisting of *bounded* sets, is forward invariant. Then we can apply Theorem 4.9 above to the random dynamical system φ generated by (18), and, via the transformation (17), get the following result to the original stochastic THC model.

Theorem 4.10. *(***Finite Degrees of Freedom***) The THC model (1)-(3) has finitely many asymptotic degrees of freedom, in the sense of having a finite set of linearly independent continuous functionals which is asymptotically determining in probability, on* $\mathcal{H}^s$ *with* $0 < s < 1/4$.

5. Conclusion

We have investigated the dynamical behavior of a random thermohaline circulation model. We have shown that the random THC model is asymptotically described by a random attractor (Theorem 4.7). And this system has finite degree of freedom in the sense of having a finite set of determining functionals in probability (Theorem 4.10).

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